

A REPRESENTATION OF FACTORIAL FUNCTION, THE NATURE OF  $\gamma$   
CONSTANT AND A WAY FOR SOLVING OF FUNCTIONAL EQUATION

$$F(x) = x \cdot F(x - 1)$$

The  $\Gamma$  function is given as recursive relation by the definition:

$$F(x) = x \cdot F(x - 1) \tag{1}$$

Derivative of the recursive relation is given as:

$$F'(x) = F(x - 1) + x \cdot F'(x - 1), \tag{2}$$

$$n!' = (n - 1)! + n \cdot (n - 1)!' \tag{3}$$

$$(n - 1)!' = (n - 2)! + (n - 1) \cdot (n - 2)!' \tag{3}$$

$\Rightarrow$

$$n!' = (n - 1)! + n \cdot ((n - 2)! + (n - 1) \cdot (n - 2)!) \tag{4}$$

$$(n - 2)!' = (n - 3)! + (n - 2) \cdot (n - 3)!' \tag{5}$$

$\Rightarrow$

$$n!' = (n - 1)! + n \cdot ((n - 2)! + (n - 1) \cdot ((n - 3)! + (n - 2) \cdot (n - 3)!)) \tag{6}$$

$$n!' = \frac{n!}{n} + \frac{n!}{n - 1} + \frac{n!}{n - 2} + n \cdot (n - 1) \cdot (n - 2) \cdot (n - 3)!' \tag{7}$$

If the recursive process is continued to  $(n - n)!'$  it will be obtained:

$$n!' = n! \cdot \left( \sum_{i=1}^n \frac{1}{i} \right) + n! \cdot C_f \tag{8}$$

Where the  $C_f$  constant is given as  $C_f = \left. \frac{\partial(n!)}{\partial n} \right|_{n=0} = -\gamma$ . Now it is possible to be obtained a series of equations that every single one is derived by a previous:

$\Rightarrow$

$$n!' = n! \cdot \left( C_f + \sum_{i=1}^n \frac{1}{i} \right) \tag{9}$$

$\Rightarrow$



$$\frac{\partial}{\partial n} \text{LOG}_e(n!) = C_f + \sum_{i=1}^n \frac{1}{i} \quad (10)$$

⇒

$$\text{LOG}_e(n!) = \int_0^n \left( C_f + \sum_{i=1}^n \frac{1}{i} \right) \cdot dn \quad (11)$$

This sum can be summarized regarding the next formula that defines the sum of geometric series:

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1} \quad (12)$$

Now, the next equations is given that may be applied for the summarization of the series:

$$\int_0^x \left( \sum_{i=0}^{n-1} x^i \right) \cdot dx = \int_0^x \frac{x^n - 1}{x - 1} \cdot dx \quad (13)$$

⇒

$$\sum_{i=0}^{n-1} \frac{x^{i+1}}{i+1} = \int_0^x \frac{x^n - 1}{x - 1} \cdot dx \quad (14)$$

⇒

$$\sum_{i=1}^n \frac{x^i}{i} = \int_0^x \frac{x^n - 1}{x - 1} \cdot dx \quad (15)$$

The final form of the formula for the series summarization is:

$$\sum_{i=1}^n \frac{1}{i} = \lim_{x \rightarrow 1} \sum_{i=1}^n \frac{x^i}{i} = \int_0^1 \frac{x^n - 1}{x - 1} \cdot dx \quad (16)$$

Now It can be derived the formula of the factorial function:

$$n! = \text{EXP} \left( \int_0^n \left( C_f + \sum_{i=1}^n \frac{1}{i} \right) \cdot dn \right) = \text{EXP} \left( C_f \cdot n + \int_0^n \left( \int_0^1 \frac{x^n - 1}{x - 1} dx \right) \cdot dn \right) \quad (17)$$

⇒

$$n! = \text{EXP}(C_f \cdot n) \cdot \text{EXP} \left( \int_0^n \left( \int_0^1 \frac{x^n - 1}{x - 1} \cdot dx \right) \cdot dn \right) \quad (18)$$

Integrals may swap the places and thus the next formula is obtained:

$$n! = \text{EXP}(C_f \cdot n) \cdot \text{EXP} \left( \int_0^1 \frac{x^n - \text{LOG}_e(x^n) - 1}{(x - 1) \cdot \text{LOG}_e(x)} \cdot dx \right) \quad (19)$$

Anent,

$$\text{LOG}_e(n!) = C_f \cdot n + \int_0^1 \frac{x^n - \text{LOG}_e(x^n) - 1}{(x - 1) \cdot \text{LOG}_e(x)} \cdot dx \quad (20)$$

Now,  $\Psi(x)$  can be defined on the next way; while the next equation is valid:

$$\frac{1}{x!} \cdot \frac{\partial(x!)}{\partial x} = C_f + \sum_{i=1}^x \frac{1}{i} = \Psi(x+1) = C_f + \int_0^1 \frac{y^x - 1}{y - 1} \cdot dy \quad (21)$$

Where is:

$$C_f = -\gamma \quad (22)$$

⇒

$$\boxed{\Psi(x) = -\gamma + \int_0^1 \frac{y^{x-1} - 1}{y - 1} \cdot dy} \quad (23)$$

Let we observe the next expression:

$$\text{LOG}_e(n!) = C_f \cdot n + \int_0^1 \frac{x^n - n \cdot \text{LOG}_e(x) - 1}{(x-1) \cdot \text{LOG}_e(x)} \cdot dx = C_f \cdot n + \int_0^1 \frac{x^n - 1}{x-1} \cdot \frac{1}{\text{LOG}_e(x)} \cdot dx \quad (24)$$

With respect to the upper expression for factorial, it can be examined the nature of the  $\gamma$  constant. If we assume that is  $n = 1$ , the solution of proper integral is irrational number. Its exponent is irrational number, too. The total solution for the situation is integer number, i.e. zero. Consequently the  $C_f$  constant must be irrational number. See the formula:

$$\text{LOG}_e(n!) = C_f \cdot n + \int_0^n \left( \int_0^1 \frac{x^n - 1}{x-1} dx \right) \cdot dn \quad (25)$$

Let is  $n = 1$ :

$$C_f = - \int_0^1 \left( \int_0^1 \frac{x^n - 1}{x-1} dx \right) \cdot dn = - \int_0^1 \frac{x - \text{LOG}_e(x) - 1}{(x-1) \cdot \text{LOG}_e(x)} \cdot dx = - \int_0^1 \left( \frac{1}{\text{LOG}_e(x)} - \frac{1}{x-1} \right) \cdot dx \quad (26)$$

⇒

$$\gamma = \int_0^1 \left( \frac{1}{\text{LOG}_e(x)} - \frac{1}{x-1} \right) \cdot dx \quad (27)$$

⇒

$$C_f = \Psi(1) = \lim_{x \rightarrow 0} \left( \text{Ei}(\text{LOG}_e(x)) - \text{Ei}(\text{LOG}_e(1+x)) - \text{LOG}_e\left(1 - \frac{1}{x}\right) \right) \quad (28)$$

⇒

$$\gamma = \lim_{x \rightarrow 0} \left( \text{Ei}(\text{LOG}_e(x)) - \text{Ei}(\text{LOG}_e(1+x)) - \text{LOG}_e\left(1 - \frac{1}{x}\right) \right) \quad (29)$$

The next numeric formula for factorial computing is derived by the equations (19) and (26):

$$n! = \text{EXP} \left( \int_0^1 \frac{x^n - n \cdot (x-1) - 1}{(x-1) \cdot \text{LOG}_e(x)} \cdot dx \right) \quad (30)$$

The next formula for computing of the  $\Psi$  is obtained with respect to the formulas (21) and (30):

$$\Psi(n) = \int_0^1 \left( \frac{x^{n-1}}{x-1} - \frac{1}{\text{LOG}_e(x)} \right) \cdot dx \quad (31)$$

By the formula (30) it can be computed the numerical solution of  $(\sqrt{-1})!$  :

$$i! = 0.49801566811835604271 - 0.15494982830181068513 \cdot i \quad (32)$$

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